ORTHIC CURVES

by

Charles Edward Brooks

O R T H I C   C U R V E S

or

ALGEBRAIC CURVES which satisfy LAPLACE'S EQUATION

in

TWO DIMENSIONS

A dissertation submitted to the Board of University Studies of the Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy

by

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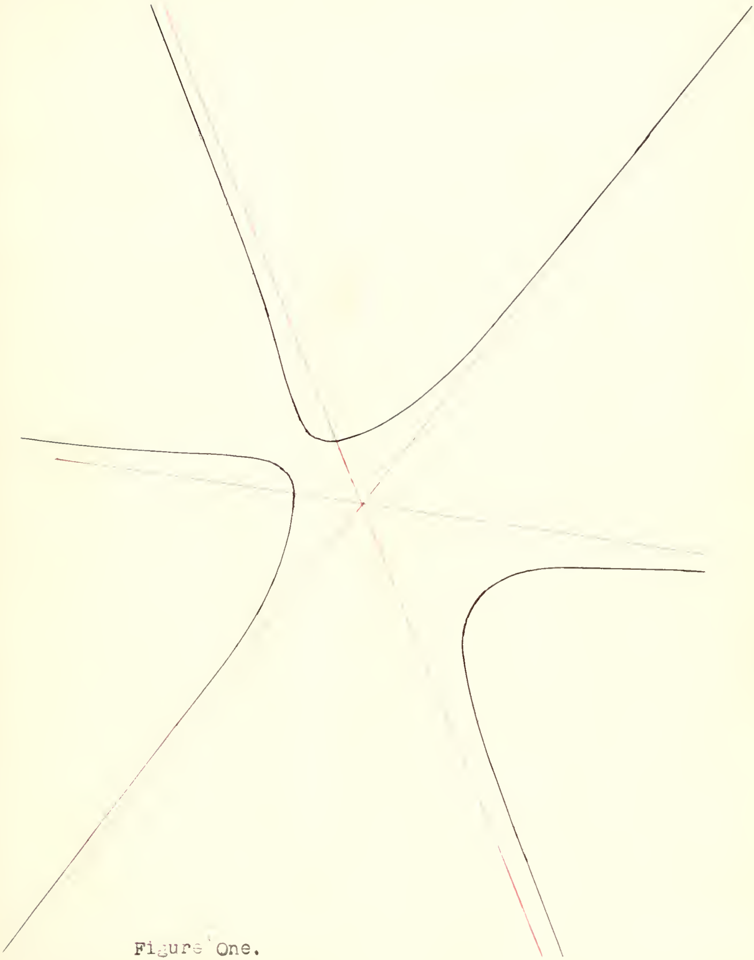

Figure One.

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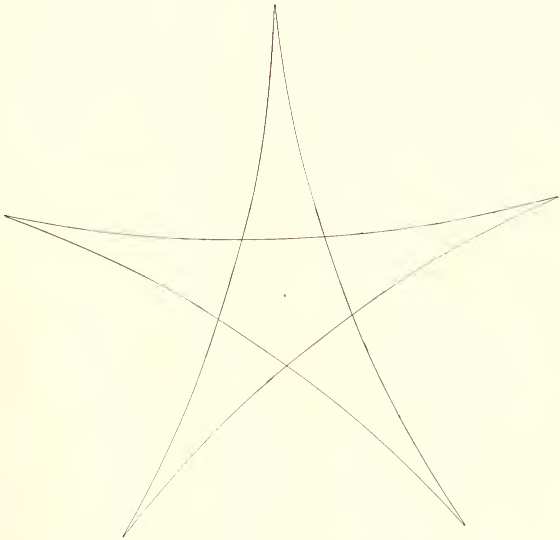

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PART ONE

The ORTHIC CUBIC CURVE

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I propose a study of the metrical properties of algebraic plane curves which are apolar, or, as it is sometimes called, harmonic, with the absolute conic at infinity. If we disregard the right line, the simplest orthic curve is the equilateral (conic) hyperbola, and the name equilateral hyperbola is sometimes extended to orthic curves of higher order. Doctor Holzmüller[1] who devotes a section to curves of this kind, calls them hyperbolas; and M. Lucas[2] calls them "stelloides".

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(1) EINFUHRUNG in die THEORIE DER ISOGONALEN VERWANDSCHAFTEN und der CONFORMEN ABBILDUNGEN, Gustav Holzmüller, Leipzig, 1882. p. [illegible]02....

(2) GEOMETRIE des POLYNOMES. Felix Lucas. Journal de L'Ecole Polytechnique, 1879. t.XXVIII.

M. Paul Serret, in a series of three papers in <u>Comptes Rendus</u>[1] uses the word "<u>équilatère</u>" for a curve with asymptotes concurrent ~~congruent~~ and parallel to the sides of a regular polygon. It seems advisable to follow M. Serret's usage, and to denote such a curve by the name equilateral, using another term to express apolarity with the absolute. For this purpose I have adopted the word orthic.

If we use Cartesian coordinates, a curve,

$$U(X, Y) = 0,$$

is apolar with the absolute conic,

$$\xi^2 + \eta^2 = 0,$$

if

$$\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} = 0.$$

In other words, an orthic curve is one which satisfies Laplace's equation in two dimensions.

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(1) Comptes Rendus, 1895., t.121. Sur les hyperboles équilatères d'ordre quelconque. p.340.

Sur les faisceaus reguliers et les équilatères d'ordre n. p.372.

Sur les équilatères comprises dans les equations

$$0 = \sum_1^{2n-2} l_1 T_1^n \equiv H_n,$$

$$0 = \sum_1^{2n-1} l_1 T_1^n \equiv H_n + \lambda H_1'$$

p.438.

PART ONE

The ORTHIC CUBIC CURVE

I.

In the analysis which may be required, I shall employ conjugate coordinates. x, $\bar{x}$. which may be defined as follows: If X and Y are the rectangular Cartesian coordinates of any point. the conjugate coordinates of that point are

$$x = X + iY, \qquad \bar{x} = X - iY,$$

when the origin is retained, and the axis of X is chosen as axis of reals, or base line. It is sometimes convenient to think of x as the vector from the origin to the point, and of $\bar{x}$ as the reflection of that vector in the base line. If x, $\bar{x}$, is a real point of the plane, not on the base line $x - \bar{x} = 0$, x and $\bar{x}$ are conjugate complex numbers. Since if one of the coordinates of a point is known, the other is immediately obtainable, we usually name a point by giving but one of its coordinates. It is convenient to reserve the letters t and τ for points of the unit circle,

$$x\bar{x} = 1.$$

Now Laplace's equation,

$$\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} = 0,$$

when applied to a function of x and $\bar{x}$, becomes

$$\frac{\partial^2 U(x\,\bar{x})}{\partial x \cdot \partial \bar{x}} = 0$$

It follows that:

The necessary and sufficient condition that a curve be orthic is that its equation in conjugate coordinates contain no product-term.

II.

Let us now proceed to the study of the orthic curve of the third order. I shall obtain the equation of an orthic cubic in a way which will suggest immediately a method for the construction of points of the curve.

The path of a point which moves in such a way that it preserves a constant orientation from three fixed points is an orthic cubic curve.

If x is the moving point, and the three fixed points are α, β, γ, then the sum of the amplitudes of the strokes which connect x with α, β, and γ must remain constant. That is, we must have

$$(x-\alpha)(x-\beta)(x-\gamma)=\rho\,T_1 .$$

If the curve is to be real, the conjugate relation,

$$(\bar{x}-\bar{\alpha})(\bar{x}-\bar{\beta})(\bar{x}-\bar{\gamma})=\rho\,T_1^{-1},$$

must hold simultaneously. The equation of the curve is obtained by eliminating the parameter ρ between these. It is

$$x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\alpha\gamma)x - \alpha\beta\gamma$$
$$= T_1^2\{\bar{x}^3 - (\bar{\alpha}+\bar{\beta}+\bar{\gamma})\bar{x}^2 + (\bar{\alpha}\bar{\beta}+\bar{\beta}\bar{\gamma}+\bar{\gamma}\bar{\alpha})\bar{x} - \bar{\alpha}\bar{\beta}\bar{\gamma}\}.$$

This is the most general equation of the third degree which we can have, without introducing the product. As a consequence it represents a perfectly general orthic cubic.

If we transform to

$$x = \tfrac{1}{3}(\alpha+\beta+\gamma),$$

the centroid of $\alpha\beta\gamma$, as a new origin, and so choose the base line that T_1^2 is real, the equation takes the form

$$x^3 + a_0 x + a_1 + \bar{a}_0\bar{x} + \bar{x}^3 = 0.$$

The equation of any orthic cubic can be brought to this form. The three points, α, β, and γ are on the curve, and form what it is convenient to call a triad of the curve.

III.

Consider the orthic cubic,

$$x^3 - S_1 x^2 + S_2 x - S_3 = T_1^2\{\bar{x}^3 - \bar{S}_1\bar{x}^2 + \bar{S}_2\bar{x} - \bar{S}_3\},$$

where the S's are the elementary symmetric functions of α, β, γ.
The approximation at infinity,

$$(x - \tfrac{1}{3}S_1)^3 - T_1^2(\bar{x} - \tfrac{1}{3}\bar{S}_1)^3 = 0,$$

makes both the square and the cube terms vanish, and therefore represents the asymptotes. The factors of this are:

$$x - \tfrac{1}{3}S_1 - \sqrt[3]{T_1^2}\,(\bar{x} - \tfrac{1}{3}\bar{S}_1) = 0,$$
$$x - \tfrac{1}{3}S_1 - \omega\sqrt[3]{T_1^2}\,(\bar{x} - \tfrac{1}{3}\bar{S}_1) = 0,$$

$$x - \tfrac{1}{3} s_1 - \omega^2 \cdot \sqrt[3]{T_1^{\,2}}\,(\bar{x} - \tfrac{1}{3}\bar{s}_1) = 0,$$

where $\omega^3 = 1$.

These three lines meet at the point

$$x = \tfrac{1}{3}(\alpha + \beta + \gamma),$$

which we may call the centre of the curve. We notice that:

<u>The centre of the orthic cubic is the centroid of the triad.</u>

The clinants of the asymptotes are $\sqrt[3]{T_1^2}, \omega\sqrt[3]{T_1^2}, \omega^2\sqrt[3]{T_1^2}$. They differ only by the constant factor ω. Now we know that multiplying the clinant of a line by ω is equivalent to turning the line through an angle $\frac{2\pi}{3}$. A rotation $\frac{2\pi}{3}$ about the centre sends each asymptote into another. It follows that the asymptotes of an orthic cubic are concurrent and parallel to the sides of a regular triangle. M. Serret[1] calls such a figure of equally inclined lines which meet in a point a regular pencil, and a curve with asymptotes forming a regular pencil he calls an "<u>équilatère</u>". Now any cubic curve, the asymptotes of which form a regular pencil, can be brought to the form:

$$x^3 + a_c x + a_1 + \bar{a}_0 \bar{x} + \bar{x}^3 = 0$$

in which we recognize it as orthic. It follows that:

<u>The orthic cubic and the equilateral of order three</u>

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(1) Comptes Rendus, Sur les hyperboles équilatères d'ordre quelconque. 1895. t.121., p,340.

are identical.

The relation

$$(x-\alpha)(x-\beta)(x-\gamma) = \rho\tau_1 = z$$

may be regarded as mapping a line through the origin in the z plane,

$$z - \tau_1^2 \bar{z} = 0,$$

into the orthic cubic. We are thus able to identify the latter with the curves discussed by Holzmüller[1] and by Lucas[2].

IV.

A figure of the orthic cubic may be obtained without difficulty by constructing points of the curve. In order to show how this may be done, it is necessary to prove the following lemma.

Elements of the pencil of orthic, or equilateral, hyperbolas which have the stroke $\beta\gamma$ as a diameter, intersect corresponding elements of the pencil of lines through α on an orthic cubic of which $\alpha\beta\gamma$ is a triad.

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(1) Holzmüller, Conformen Abbildungen. p. 205.

(2) Lucas, Géométrie des Polynomes. Journal de l'Ecole Polytechnique, t.XXVIII, p. 23.

For the line through α,

$$x - \alpha = \rho T',$$

and the equilateral hyperbola on $\beta\gamma$ as a diameter,

$$(x - \beta)(x - \gamma) = \rho T'',$$

intersect on the orthic cubic,

$$(x - \alpha)(x - \beta)(x - \gamma) = \rho T_1,$$

if

$$T' T'' = T_1 .$$

If the two pencils are given, it is only necessary to

~~to~~ pair off lines and curves according to the relation

$$T'T'' = T_1,$$

and to mark intersections. These will be points of the curve.

I have had constructed a very simple instrument for drawing the equilateral hyperbolas required in the construction given above. Two toothed wheels of equal diameters are attached beneath the drawing board in such a way that their teeth engage. The axles are perpendicular to the board and come through it at β and γ. The axles, which turn with the wheels, carry long hands or pointers which sweep over the board. On account of the cogs, the wheels ~~have to~~ can turn only through equal and opposite angles. As a consequence, x, the intersection of the hands, has a constant orientation, from β and γ, and in fact, generates the orthic curve of the second order given by

$$(x-\beta)(x-\gamma) = \rho T'.$$

But this is the hyperbola required. The accompanying figure[1] was drawn with the aid of this device. The actual labor of drawing is lessened by the fact that the centre and asymptotes are know. The centre is the centroid of $\alpha\beta\gamma$ and the asymptotes have the amplitudes of the cube roots of $-T_1^2$.

[1] Figure One

V.

A mechanism which will actually draw an orthic cubic is very much to be desired. One might be made in some such way as the following. Suppose three hands, like those described above (IV), to be pivoted at α, β, and γ. Let them be held together in such a way that while each is free to move along the others. ~~the~~ they must always meet in a point, which is to be the tracing point. Each hand is to receive its motion from a cord wound about a bobbin on its axle. The bobbins are to be equal in diameter. The cords pass through conveniently placed pulleys and are kept tight and vertical by small equal weights at their ends. Consider, to fix ideas, those three weights which by their descent give the hands positive rotation. If, now, the tracing point be moved along an orthic cubic which has $\alpha\beta\gamma$ for a fundamental triad, the total turning of the bobbins will be zero, and as a consequence the total descent of the weights will be zero. Conversely, if we can move these vertically and in such a way that the total descent will be zero, the tracing point can move only along an orthic cubic. This ~~desirable~~ result will be obtained if the centre of gravity of the three weights can be kept fixed. It will not do, however, to connect the three weights by a rigid triangle pivoted at its centre of gravity.

for then they will not move vertically. But since a parallel projection does not alter the centroid of a set of points, the desired result will be attained if the weights are constrained to vertical motion by some kind of guides. and are kept in a plane which always passes through the centre of gravity of one position of the weights.

VI.

Consider the general orthic cubic given by

$$x^3 - a_0 x^2 + a_1 x - a_2 + a_3 \bar{x} - a_4 \bar{x}^2 + a_5 \bar{x}^3 = 0 .$$

It cuts the unit circle,

$$x \bar{x} = 1 ,$$

in six points, the roots of

$$x^6 - a_0 x^5 + a_1 x^4 - a_2 x^3 + a_3 x^2 - a_4 x + a_5 = 0 .$$

If we want the cubic to meet the circle in six given points, say, $T_1, T_2, \cdots T_6$, then this equation must be identical with

$$x^6 - S_1 x^5 + S_2 x^4 - S_3 x^3 + S_4 x^2 - S_5 x + S_6 = 0,$$

in which the S's stand for the elementary symmetrical combinations of the six T's. This requires

$$a_0 = S_1 , \quad a_1 = S_2 , \quad a_2 = S_3 ,$$

$$a_3 = S_4 , \quad a_4 = S_5 , \quad a_5 = S_6 .$$

The coefficients of the cubic equation are then precisely determined. with the result that: <u>But one orthic cubic can constructed through any six points of a circle.</u>

It remains for us to show that one such curve can always be drawn: that is, that the equation

$$x^3 - S_1 x^2 + S_2 x - S_3 + S_4 \bar{x} - S_5 \bar{x}^2 + S_6 \bar{x}^3 = 0$$

always represents a real curve. If we so choose the base line that $S_6 = 1$, then we have

$$\bar{S}_i = S_{6-i} S_6^{-1} = S_{6-i} ,$$

and the equation takes the form

$$x^3 - s_1 x^2 + s_2 x - s_3 + \bar{s}_2 \bar{x} - \bar{s}_1 \bar{x}^2 + \bar{x}^3 = 0.$$

which is, obviously, self-conjugate, and is, therefore, satisfied by the coordinates of real points. As a result:

An orthic cubic can always be drawn through six points of a circle. It is then determined uniquely.

VII.

When the orthic cubic is referred to the six points in which it cuts the unit circle, the equations of the asymptotes take the form

$$x - \tfrac{1}{3} s_1 = (-s_6)^{\frac{1}{3}} (\bar{x} - \tfrac{1}{3} s_5 s_6^{-1}).$$

These three lines meet at

$$x = \tfrac{1}{3} s_1 ,$$

the centre. This point, the origin, and the point which is the centroid of the six points on the circle lie on a line; and the latter point is midway between the other two. This leads to the interesting fact that:

The centroid, of the six points in which any circle meets an orthic cubic bisects the stroke from the centre of the curve to the centre of that circle.

VIII.

We spoke of the three points α, β, γ, which have the same orientation from every point of the curve, as a

triad of the curve. Let us see how many such triads there are, and how they are arranged. The relation

$$(x-\alpha)(x-\beta)(x-\gamma)=z$$

may be regarded as establishing a correspondence between points x in one plane and points z in another plane, in such a way that if z describes a line ξ, through the origin, the point x generates an orthic cubic on $\alpha\,\beta\,\gamma$ as a ~~fundamental~~ triad. To every position of z on the director line ξ, there correspond three points in the x-plane. I shall show that each such set of three points is a triad. Write

$$F(x)=(x-\alpha)(x-\beta)(x-\gamma).$$

Then, if x_1, x_2, x_3 are the three points which correspond to z,

$$F(x)-z=(x-x_1)(x-x_2)(x-x_3).$$

And also

$$F(x)-z'=(x-x_1')(x-x_2')(x-x_3').$$

Now this relation is satisfied by x_1, or x_2, or x_3 ______ read

$$F(x_1)-z'=(x_1-x_1')(x_1-x_2')(x_1-x_3')=z-z'.$$ read

Since $z-z'$ is a point of the director line, it follows that the three points x_1', x_2', x_3', which correspond to any point z' of the director line have the same orientation from every point of the curve. We conclude that

TP <u>To every point of the director line corresponds a triad; all the points of the curve have the same orientation from any triad, and all the triads of the curve have the same orientation from any point of the curve.</u>

IX.

We seek the points of a triad which correspond to a given point $z^{(1)}$. The map equation can be brought to the form

$$x^3 - 3x = 2z,$$

by choosing the centre of the curve as new origin and making a suitable choice of the unit stroke. We see at once that the sum of the x's for a given z is zero, In other words. <u>The centroid of any triad is the centre of the cubic.</u>

Making use of the method known as Cardan's solution, put

$$x = \mu t + v .$$

where μ is real.
Then

$$x^3 - 3x = 2z$$

becomes

$$\mu^3 t^3 + v^3 + 3\mu^2 t^2 v + 3\mu t v^2 - 3(\mu t + v) = 2z.$$

And we have as two relations between μt and v,

$$2z = \mu^3 t^3 + v^3$$

and

$$(\mu t + v)(\mu t v - 1) = 0$$

When z is zero, the values of x are $\pm\sqrt{3}$ and 0 ; and when z is not zero, we must have

$$v = \frac{1}{\mu t}$$

This leads to the expression of x and z in terms of μt as follows :

(1) Harkness and Morley, A Treatise on the Theory of Functions p. 39.

$$x = \mu t + \frac{1}{\mu t},$$
$$2z = \mu^3 t^3 + \frac{1}{\mu^3 t^3}.$$

Now if we assign any value to μ and let t run around the unit circle, x describes an ellipse with foci at $x = +2$ and $x = -2$. But at the same time, z also describes an ellipse with its foci at $z = +1$ and $z = -1$. These two ellipses are related in such a way that a point z on one of them is correlated by the equation

$$x^3 - 3x = 2z$$

with three points on the other. Now the foci of both these ellipses are independent of the particular value of μ selected; it follows that, if we assign successive values to μ, we shall obtain in each plane a system of confocal ellipses of such a sort that the equation

$$x^3 - 3x = 2z$$

~~xxxx~~ establishes a one to one correspondence between them. In each plane the origin is the centre of all the ellipses. Applying this scheme to the case in hand, we see that a triad must be inscribed in one of the ellipses in the x plane. But the centroid of the triad is the centre of the ellipse; so the ellipse must be the ~~minimum~~ circumscribed ellipse of least area of that triad. We may say then, that:

<u>The triads of the orthic cubic are cut out of the curve by a particular system of confocal ellipses; and each ellipse is the ~~minimum~~ circumscribed ellipse of least area of the triad</u>

on it.

X.

If we examine the equation

$$x^3 - 3x = 2z$$

for equal roots, we find that the double points of the x plane are at $x = +1$ and at $x = -1$. These values of x correspond to the branch points in the z-plane, $z = +1$ and $z = -1$

Let us for a moment, replace the z-plane by a three sheeted Riemann surface. All three sheets must hang together at infinity; and two sheets at each of the branch points. Let the first and second sheets be connected by a bridge along the base line from $+1$ to infinity, and the second and third sheets be similarly connected by a bridge along the real axis from -1 to infinity.

Select on this surface any large ellipse with foci at the branch points, and any line as a director line. Now consider the contour obtained by starting from a point of this inside the ellipse, going thence along the line to meet the ellipse, along an arc of the ellipse to meet the line, and then along the line to the point of departure.

We can choose this path in such a way that one of the following three cases must arise:

(1). <u>The contour passes through a branch point.</u>

(2). *The contour surrounds one branch point.*

(3). *The contour surrounds no branch point.*

In case (1), we know that the cubic must have a node. In case (2), by going three times around we can pass continuously through every sheet of the Riemann surface, and therefore through every value of x. Or, thinking again of the x-plane, we have a unicursal boundary. Now it happens that the ellipse we choose maps into one, and not three ellipses on the x-plane. If we imagine this to expand indefinitely, we shall have to consider the boundary as our orthic cubic. It follows at once that: *The orthic cubic which corresponds to a line which does not pass between the branch points is unipartite.*

If the contour includes one branch point, and therefore crosses one bridge of the Riemann surface, we must go along two unconnected curves to reach all the values of x. When these two curves are spread on the x-plane, they lead at once to the conclusion that: *The orthic cubic which corresponds to a line which passes between the branch points is a bipartite curve.*

Let us turn our attention again to the two planes connected by the relation

$$x^3 - 3x = 2z.$$

We notice that while the ellipses in the z- plane have their foci at the branch points, the foci of the corresponding system of ellipses are not the double points of the x-plane, but are the points $x = 2$ and $x = -2$, each of which, with one of the double points counted twice, forms a triad.

As a rule there are two triads of the curve on each ellipse, corresponding to the two points in which the director line cuts an ellipse of the system in the z-plane. But unless the line go between the branch points it will be tangent to one ellipse; consequently, two triads will coincide, and the cubic will be tangent at three places to one of the ellipses of the system. No part of the cubic can be inside of that ellipse.

When μ is 1, the two ellipses degenerate into two segments,

$$x = t + t^{-1}, \quad \text{on} \quad \overline{2, -2},$$

$$2z = t^3 + t^{-3}, \quad \text{on} \quad \overline{1, -1}.$$

If the line pass between the branch points, and so cut the segment $\overline{1, -1}$, two triads again coincide, but in

this case the three points lie on a line, and we do not have the triply tangent ellipse.

When the line ξ cuts the axis of imaginaries,

$$x + \bar{x} = 0,$$

we have

$$z = \rho\, \varepsilon^{\frac{\pi i}{2}}$$

and

$$t^3 = \rho'\, \varepsilon^{\frac{\pi i}{2}}$$

It follows that $\operatorname{am} t = \pi/6$, and so ωt is the reflection of t in the axis of imaginaries and $\omega^2 t$ is a pure imaginary. Then, since we know that

$$x_1 = \mu t + \frac{1}{\mu t}$$
$$x_2 = \omega \mu t + \frac{1}{\omega \mu t}$$
$$x_3 = \omega^2 \mu t + \frac{1}{\omega^2 \mu t}$$

we see that x_1 is the reflection of x_2 in the line $x + \bar{x} = 0$

and that x_3 is on that line. It follows that the triangle $x_1 x_2 x_3$ is isosceles and that its base $x_1 x_2$ is parallel to the real axis. There is again an isosceles triangle when t^3 is real. This triangle has its vertex on the axis of reals and its base perpendicular to that axis. Forming the discriminant of the quadratic in $\mu^3 t^3$

$$z^2 - 4$$

we see that t^3 is real when $z > \pm 2$. In other words,

if the director line ξ cut the axis of reals, but not between the branch points, we have such an isosceles triangle.

From the above considerations, we see that if the director line is either of the axes,

$$x + \bar{x} = 0, \quad x - \bar{x} = 0,$$

then one branch of the orthic cubic must be a right line: the remaining portion of the curve must then be an ordinary hyperbola, and the inclination of its asymptotes must be either $\pi/3$ or $\frac{2\pi}{3}$. The first value refers to the case when the director line is the axis of imaginaries, ~~and the~~ and the second to the case when it is the axis of reals.

XII.

Suppose we put a circle through the three points of a triad and ask: Where are the remaining three points in which it cuts the cubic? For convenience, let three points of the unit circle be taken as a triad. The cubic is then

$$(x-t_1)(x-t_2)(x-t_3) = \tau(\bar{x}-t_1^{-1})(\bar{x}-t_2^{-1})(\bar{x}-t_3^{-1}).$$

On eliminating x from this and the equation of the circle,

$$x\bar{x} = 1,$$

we obtain

$$(x-t_1)(x-t_2)(x-t_3)=\frac{\tau_1^2(t_1-x)(t_2-x)(t_3-x)}{t_1t_2t_3x^3}$$

or

$$x^3=\frac{-\tau_1^2}{t_1t_2t_3}$$

as the equation of the three points sought. The roots of this, $x_1=k$, $x_2=\omega k$, $x_3=\omega^2 k$, are the coordinates of the vertices of an equilateral triangle. As there is no restriction in taking the triad on the unit circle, we have the following theorem:

<u>If a circle cut an orthic cubic in a triad, then the two curves have three other intersections, which form an equilateral triangle.</u>

XIII.

We have seen that the relation

$$(x-\alpha)(x-\beta)(x-\gamma)=z$$

maps a line ~~three~~ through the origin into an orthic cubic which has $\alpha\beta\gamma$ as a triad. It must then map all the lines through the origin into a single infinity of orthic curves[(1)] which have the common triad $\alpha\beta\gamma$.

If we regard τ as a parameter, we may say that

$$(x-\alpha)(x-\beta)(x-\gamma)=\tau(\bar{x}-\bar{\alpha})(\bar{x}-\bar{\beta})(\bar{x}-\bar{\gamma})$$

is the equation of the pencil of orthic cubics which have the triad α,β,γ. It will be convenient to give a pencil of this sort some name: let us refer to it as a <u>central</u>

(1) Felix Lucas, Journal de l'Ecole Polytechnique, XXVIII. p 21.

pencil, noting for our justification that the centroid of the triad is the centre of every curve of the pencil.

If there were any real point, other than α, β, or γ on two curves of this pencil, it would map into a real point of the z-plane, not the origin, which would be on two of the lines through the origin. As this is manifestly impossible, it follows that: Two orthic cubics which have a triad in common, have no other real intersection.

Now we know that two cubics intersect in nine points, and that if the curves given by the equation

$$(x-\alpha)(x-\beta)(x-\gamma) = \tau(\bar{x}-\bar{\alpha})(\bar{x}-\bar{\beta})(\bar{x}-\bar{\gamma})$$

really constitute a pencil, there must be six imaginary points, the coordinates of which satisfy the equation, whatever the value of τ. Let us form the following table of coordinates. The real intersections are

$$x_1 = \alpha, \qquad \bar{x}_1 = \bar{\alpha},$$
$$x_2 = \beta, \qquad \bar{x}_2 = \bar{\beta},$$
$$x_3 = \gamma, \qquad \bar{x}_3 = \bar{\gamma}.$$

It is evident that each of the following points,

$$x_4 = \alpha, \qquad \bar{x}_4 = \bar{\beta},$$
$$x_5 = \alpha, \qquad \bar{x}_5 = \bar{\gamma},$$
$$x_6 = \beta, \qquad \bar{x}_6 = \bar{\alpha},$$

$$x_7 = \beta, \qquad \bar{x}_7 = \bar{\gamma},$$
$$x_8 = \gamma, \qquad \bar{x}_8 = \bar{\alpha},$$
$$x_9 = \gamma, \qquad \bar{x}_9 = \bar{\beta},$$

satisfies the equation, independently of τ. These points, the six imaginary intersections of the pencil, are the antipoints[1] obtained by selecting pairs in all possible ways from α, β, γ.

The figure of nine points in which two orthic cubics intersect may be regarded as an extension of the orthocentric four point determined by two equilateral hyperbolas. It is convenient to extend the term *orthocentric* to such a figure. Resuming the results obtained above, we have:

When three of the points of an orthocentric nine-point are a triad of ~~xxx~~ *the orthic curves through* ~~them~~ *the nine points, the remaining six points are imaginary, and are the antipoints of the three real points. The centroid of the nine points is the centre of every orthic cubic through them.*

It is convenient to speak of a set of orthocentric

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(1) Cayley. Collected Mathematical Papers, volume VI., p. 499.

points determined by a central pencil as a central set.

Since any three points determine a pencil of orthic cubics of which they are a triad, any three points, with all their antipoints, form a central orthocentric nine-point.

XIV.

We shall now attack the problem of finding the foci of the orthic cubic. Let us begin with ~~As a preliminary~~, a few words as to the way in which the foci of a curve appear in analysis with conjugate coordinates ~~may not be out of place.~~ The focus of a curve is the intersection of a tangent from one circular point with a tangent from the other circular point. In other words, if the circular rays from a point are tangent to a curve, that point is a focus of the curve. Now the equation of the circular rays from a point $\alpha,\bar{\alpha}$, is

$$(x-\alpha)(\bar{x}-\bar{\alpha})=0.$$

Therefore one of the lines is

$$x-\alpha=0,$$

and the other is

$$\bar{x}-\bar{\alpha}=0.$$

Suppose the equation of the curve is

$$f(x\,\bar{x})=0.$$

Now if the circular ray

$$x-\alpha=0$$

is tangent to the curve, then

$$f(\alpha\,\bar{x})=0,$$

the eliminant of x between these two, will have equal roots. But since the equation of a real curve must be

self-conjugate. if this has two coincident roots then

$$F(\bar{x}\ x) = 0$$

must also have, and the point $\alpha, \bar{\alpha}$, is a focus. It follows that to find the foci of a curve, ~~having known its equation in conjugate coordinates~~, we have merely to find those values of x which make two values of $\bar{x}$ coincide. They are the vectors ~~coordinates~~ of the foci. Let us apply this method to the orthic cubic. The equation may be taken in the form

$$x^3 - 3x = 2z = a_0 + \lambda a_1$$

where λ is a real parameter and the director line is

$$a_0 + \lambda a_1 = 2z, \quad \bar{a}_0 + \lambda \bar{a}_1 = 2\bar{z}.$$

These relations imply the conjugate expression

$$\bar{x}^3 - 3\bar{x} = 2\bar{z} = \bar{a}_0 + \lambda \bar{a}_1.$$

Two values of $\bar{x}$ become equal when $D_{\bar{x}}\bar{z} = 0$, i.e. when

$$\bar{x}^2 - 1 = 0$$

or

$$\bar{x} = \pm 1.$$

These values of $\bar{x}$ occur when

$$\bar{a}_0 + \lambda \bar{a}_1 = \pm 2,$$

or

$$\lambda = \frac{-\bar{a}_0 \pm 2}{\bar{a}_1}.$$

Either of these values of λ when substituted in

$$x^3 - 3x = a_0 + \lambda a_1$$

gives three points which are foci of the cubic.

There are, in general, six real foci, which fall into two sets of three. Each set of three corresponds to a single point of the z-plane and is therefore a maximum inscribed triangle of one of the ellipses described above.

XV.

If we eliminate the parameter between

$$2z = a_0 + \lambda a_1$$

and

$$2\bar{z} = \bar{a}_0 + \lambda \bar{a}_1,$$

we get the equation of the line ξ,

$$\bar{a}_1 z - a_1 \bar{z} = a_0 \bar{a}_1 - a_1 \bar{a}_0 .$$

Now suppose, for a moment, that this line does not contain either of the branch points $z = \pm 1$. Then if we put $\bar{z} = \pm 1$ in the equation of the line, and solve for z we get a value which is not the conjugate of z, but is the ~~z-coordinate~~ vector of the reflection of the point $z = \pm 1$ in the line considered. The three points in the x-plane got by putting $\lambda = \frac{-\bar{a}_0 \pm 2}{\bar{a}_1}$ in the equation

$$x^3 - 3x = 2z$$

are the points mapped in the z-plane by the reflection of $z = \pm 1$ in the line ξ. It follows that

The real foci of the orthic cubic which corresponds to a given line are the six points which correspond to the reflections in that line of the branch points.

If the director line pass through one of the branch points. (i.e.. if $\frac{-\bar{a}_0 \pm i}{a_1}$ is real), two foci coincide to form the node, and the remaining one of that set of three is on the curve. One who looks at the matter from the point of view of the Riemann surface might be surprised that a branch point is to be reflected in the line in each sheet of the surface, and not in the two sheets alone which it connects. A moments consideration will show that whether or not two $\bar{x}$'s coincide depends on λ alone, and that either of three values of x gives λ a particular value. It is clear that the reflection must be in every sheet of the surface.

In general, the orthic cubic is of class six. Since it cuts the line at infinity in three points apolar with the circular points, it cannot contain one of the circular points except as a point of inflection. There should be, therefore, six tangents from each of the circular points and, consequently, thirty-six foci. The thirty foci still to be accounted for are the antipoints(1) of the real foci, paired in all ways. When the cubic has a node, it is of class four, and has but four real foci. The node therefore takes the place of the two foci which coincide there.

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(1) Salmon, Higher Plane Curves, third edition. p., 122.

The circular rays

$$x - \alpha_1 = 0$$

and

$$\overline{x} - \overline{\alpha}_2 = 0$$

meet at $\alpha_1 \overline{\alpha}_2$. So we may represent the thirty-six foci of an orthic cubic by the scheme of coordinates: $\alpha_i, \overline{\alpha}_j$. where i and j run from 1 to 6. It follows that the centroid of the whole thirty-six is the centroid of the six real ones, that is, the centre of the curve. Consider any selection of three foci. All their antipoints are foci, and the nine points together make up a central orthocentric set.

XVI.

The foci of all the orthic cubics which have a common triad $\alpha\beta\gamma$ lie on two cassinoids which have their foci at α, β, and γ; and are orthogonal to the orthic curves.

We know that these cubics correspond to all the lines through a point, and that their foci correspond to the reflections of the branch points in those lines. Now the reflections of a fixed point in all the lines through a second point lie on a circle which goes through the first point. Accordingly, the foci of the cubics will lie on the curves which are the maps in the x-plane of two concentric circles in the z-plane. The centre of these circles maps into the triad common to all the cubics.

and the circles themselves map into two cassinoids of the sixth order about the triad. as M. Lucas has shown.(1) Each of the circles goes through one of the branch points, and therefore each of the cassinoids must have a node. If the point which corresponds to the triad $\alpha\beta\gamma$ is equidistant from the two branch points, the two circles and also the two cassinoids, coincide. In this case the the latter has two double points.

The lines which correspond to the cubics are all perpendicular to the circles which correspond to the cassinoids; and so, by the principle of orthogonality, the cassinoids are orthogonal trajectories of the cubics of the pencil.

XVII.

I shall close this study of the metrical properties of the orthic curve of the third order by showing that from the point of view of projective geometry the orthic cubic is really a general cubic. Any proper plane curve of the third order can be projected into an orthic curve.

We know that the points of contact of three of the six tangents to a cubic curve from any point of its Hessian lie in a line. Now these three points, ~~if real~~, considered

(1) Geometrie des Polynomes. Felix Lucas, Journal de l'Ecole Polytechnique, t.,XXVIII, p.,5.

as a binary cubic, have an ~~imaginary~~ Hessian pair. If this pair of points be projected to the circular points at infinity, the three tangents become equally inclined asymptotes, and they continue to meet in a point. The cubic curve is then orthic, and the transformation is then accomplished. This projection only requires two points to go into two given points, and can, therefore, always be made. In projective geometry, the orthic cubic is any proper plane cubic.

As an illustration of the way in which information about the orthic cubic applys to cubic curves in general, let us see what the characteristic property that the asymptotes are concurrent and equally inclined.(i.e., apolar with I and J.) means. The circular points, I and J are a pair of points apolar with the curve. Their join, the line at infinity, meets the curve in three points such that the tangents at these points meet in a point, C, of the Hessian. Now we know(1) that such a line meets the Hessian in the point which corresponds to C. This leads to the theorem that:

The line joining two points apolar with a cubic curve meets the cubic in three points, the tangents at which

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(1) Salmon, Higher Plane Curves, third edition, articles 70 and 175.

meet in a point of the Hessian, and are apolar with the two points apolar with the curve.(1)

The line joining two points apolar with a cubic curve, and a tangent to the cubic at a point of this line, meet the Hessian of the given cubic in corresponding points.

A more novel result is the following. We have seen, (XIV. p.28.), that the foci of an orthic cubic fall into two sets of three, in such a way that the two sets are triangles of maximum area inscribed in two confocal ellipses. Now if we consider tangents from I and J instead of foci, we have the following theorem.

If a and b are a pair of points apolar with a cubic curve, then the tangents from either of these points, say a, fall into two sets of three in such a way that the line ab has the same polar pair of lines as to each set of three.

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(1) On the Algebraic Potential Curves. Dr. Edward Kasner. Bulletins of the American Mathematical Society, June, 1901. p..393

PART TWO

ORTHIC CURVES of any ORDER

PART TWO.

Orthic Curves of any Order.

I.

In the preceding pages, we have studied the metrical properties of the orthic cubic in some detail. In the following portion of the work I shall indicate an extension of the more important results obtained in the study of the cubic to orthic curves of any order.

The general equation of the n^{th} degree between x and $\overline{x}$ contains $\frac{1}{2}n(n-1)$ product terms. If it is to represent an orthic curve the coefficients of these terms must be made zero. In other words, to make a curve of the n^{th} order orthic is equivalent to making it satisfy $\frac{1}{2}n(n-1)$ linear conditions. After this has been done there remain $2n$ degrees of freedom.

II.

The kinematical definition which we obtained for the orthic cubic may be extended to curves of any order: that is:

<u>The path of a point which moves so that its orientation from n fixed points is constant is an orthic curve of order n.</u>

If $\alpha_1, \alpha_2 \cdots \alpha_n$ are the fixed points, the condition on x is expressed by the relations

$$(x - x_1)(x - x_2)\cdots(x - x_n) = r\tau_1$$

and

$$(\bar{x} - \bar{x}_1)(\bar{x} - \bar{x}_2)\cdots(\bar{x} - \bar{x}_n) = r\tau_1^{-1}.$$

These lead to the equation of the curve,

$$x^n - S_1 x^{n-1} + S_2 x^{n-2} + \cdots S_n + \tau_1^2(\bar{S}_n \cdots \bar{S}_1 \bar{x}^{n-1} - \bar{x}^n) = 0,$$

where the S's are the elementary symmetric combinations of the α's. This is the general equation of an orthic curve. If we take $x = 1/n\, S_1$ for a new origin, and make τ_1^2 real, the equation becomes

$$x^n + a_1 x^{n-2} - a_2 x^{n-3} \cdots - \bar{a}_2 \bar{x}^{n-3} + \bar{a}_1 \bar{x}^{n-2} + \bar{x}^n = 0.$$

The asymptotes are the n equally inclined lines given by the factors of the highest terms,

$$x^n + \bar{x}^n = 0.$$

These lines all pass through the origin; it follows that the centroid of the n points $\alpha_1 \cdots$ is the centre of the curve. Since every orthic curve can be brought to the above form, we see that every orthic curve is equilateral. The converse proposition, Every equilateral is orthic, is not true. The general equation of an equilateral may be put in the form

$$x^n + a\bar{x}^n + \Phi(x\bar{x}) = 0.$$

where $\Phi(x\bar{x})$ is a perfectly general function of degree n-2. Φ Contains 1/2(n-2)(n-?) product terms, which must vanish for the curve to be orthic. To make an equilateral

curve orthic, is therefore, equivalent to making it satisfy 1/2(n-2)(n-3) linear conditions. For $n=2$ and $n=3$ this number is zero, so the equilateral conic and cubic are orthic. For the quartic, this says that to be orthic is one condition.

III.

The relation

$$(x-x_1)(x-x_2)\cdots(x-x_n)= t\,\tau_1 = z$$

may be regarded as maping a line through the origin in the z-plane into the orthic curve in the x-plane. The methods of analysis which were used, in the paragraphs referred to, in the study of the orthic cubic may be extended to any n, and lead to the following general theorems.

<u>On an orthic curve of order</u> .n <u>there is a single infinity of sets of</u> n <u>points, n-ads of the curve, from which all points of the curve have the same orientation, All the n-ads have the same orientation from any point of the curve</u> .(Part One, VIII.).

Any n points may be taken as the an n-ad of an orthic curve. If we take n points of the unit circle as an n-ad, and find the remaining intersections of the circle and the curve, we see that they are the vertices of a regular polygon. (Part One, XII.).

<u>Every circle through an n-ad of an orthic curve of</u>

order n meets the curve again in the n vertices of a regular polygon.

The centre of an orthic curve is the centroid every n-ad of the curve.

For when the equation is taken in the form

$$x^{n} + n\,x^{n-2} + \;+\; a_{n-3}\,x = z$$

the origin is the centre of the curve, and is also the centroid of the n points which correspond to a point z. This equation will have two coincident roots whenever

$$D_x z \equiv n x^{n-1} + n \cdot n-2 a_1 x^{n-3} + \cdots = 0$$

In general, this will give n-1 branch points in the z-plane. Each branch point, when reflected in the director line, gives rise to n real foci.[1] If the line $\int$ revolve about a point, each reflection generates a circle. All n-1 of these circles are concentric; and they map into n-1 cassinoids which are the loci of the foci of the curves which have the n-ad which corresponds to the centre of the system of circles. These cassinians are orthogonal trajectories of the central pencil of orthic curves. Since each of the circles must contain a branch point, each cassinian must have at least one node.

IV.

We know that we may put 2n linear conditions on an orthic curve. If we make it go through 2n points of the unit circle, its equation, expressed in terms of the points where it meets the circle, becomes

$$x^n - S_1 x^{n-1} + S_2 x^{n-2} \cdots (-1)^n S_n \pm \cdots - S_{2n-1} \bar{x}^{n-1} + S_{2n} \bar{x}^n = 0$$

where the S's are the symmetrical sums of the t's.

The centre, got by equating the $n-1^{st}$ derivitive with respect to x to zero is

$$x = \frac{1}{n} S_1$$

This is the mid-point of the stroke from the centre of the circle to the centroid of the 2n points.

The equation of an asymptote now takes the form

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(1) Part One, XIV.

$$x - {}^1\!/_n\, s_1 = \sqrt[n]{-s_{2n}}\,\left(\bar{x} - s_{2n-1} \cdot s_{2n}^{-1}\right).$$

V.

The method which I have proposed (Part One, V.) for the construction of an orthic cubic might be extended to the construction of any orthic curve. For this purpose the instrument must have n hands, moved by n weights. The centre of gravity of any number of weights could be held by joining them together in sets of three or less, and then joining again the centres of gravity of these sets. This operation could be repeated until the required number of weights is reached.

VI.

<u>The geometrical characteristics of an orthic curve of order n are that it is equilateral, and that it intersects its asymptotes in points of a second orthic curve of order n-2.</u>

For consider the orthic curve referred to the centre,

$$x^n + a_1 x^{n-2} + a_2 x^{n-3} \cdots + \bar{a}_2 \bar{x}^{n-3} + \bar{a}_1 \bar{x}^{n-2} + \bar{x}^n = 0$$

The asymptotes, which are given by

$$x^n + \bar{x}^n = 0,$$

are congruent and equally inclined, so the curve is

equilateral. The points common to the curve and its asymptotes lie on the curve

$$a_1 x^{n-2} - a_2 x^{n-3} + \cdots - \bar{a}_2 \bar{x}^{n-3} + \bar{a}_1 x^{n-2} = 0.$$

But this curve is of order n-2. and is orthic.

To require a curve to be equilateral is to impose 2n-3 conditions, and to require the curve of order n-2 along which it cuts its asymptotes to be orthic is to impose 1/2(n-2)(n-3) further conditions, in all 1/2{n(n-1)}. But 1/2{n(n-1)} is the number of conditions required to make a curve of order n orthic.

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PART THREE

PENCILS DETERMINED by two ORTHIC CURVES

and

ORTHOCENTRIC SETS OF POINTS

I.

We shall now take up the study of the pencils of curves determined by two orthic curves. The main purpose of this investigation shall be to learn what we can about the figure of n^2 points in which two orthic curves intersect. Such a figure of n^2 points we shall call an Orthocentric Set, or an Orthocentric n^2-point.

There is a well known proposition that all the equilateral hyperbolas (orthic conics) which can be circumscribed to a given triangle pass through the orthocentre of the triangle. The four points, the vertices and the orthocentre of a triangle, or, what is the same thing, the the intersections of two orthic curves of the second order, have the property that the line joining any two of them is perpendicular to the line joining the other two. The term orthocentric is applied to a set of four points related in this way. We wish to find out what metrical property distinguishes the n^2-point in which two orthic curves of order n intersect.

The first generalization which we shall make is to show that any pair of points α, β, together with their anti-points $\alpha, \overline{\beta}$ and $\beta, \overline{\alpha}$ form an ~~orthic~~ eccentric four point. α and β determine a central pencil of orthic conics

$$(x-\alpha)(x-\beta)=\tau(\overline{x}-\overline{\alpha})(\overline{x}-\overline{\beta}),$$

and the anti-points are evidently on all the curves of the pencil.

If we consider τ as a parameter in the general equation of an orthic curve,

$$(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)=\tau(\overline{x}-\overline{\alpha}_1)(\overline{x}-\overline{\alpha}_2)\cdots(\overline{x}-\overline{\alpha}_n),$$

we obtain the equation of all the curves of which $\alpha_1,\cdots\alpha_n$ is a fundamental n-ad. The points of the orthocentric n^2-point determined by this are the n real points $\alpha_1,\cdots\alpha_n$ and all their anti-points. But as the pencil is determined by the n real points, it follows that <u>Any n points, with all their anti-points, form a central orthocentric n^2-point.</u>

The centroid of the n^2-point determined by a central pencil is

$$x=\frac{1}{n^2}\sum n\alpha_1+n\alpha_2+\cdots n\alpha_n$$
$$=\frac{1}{n}(\alpha_1+\alpha_2+\cdots\alpha_n).$$

This is the centroid of the n real points, and it is also the centre of the pencil. The real and imaginary foci of any curve form an example of a central orthocentric set of points.

III

We have seen that six points of a circle determine an orthic cubic curve. If the six points are $t_1, t_2, t_3, t_4, t_5, t_6$ then, as we have seen, the equation of the orthic cubic through them is

$$x^3 - S_1 x^2 + S_2 x - S_3 + S_4 \bar{x} - S_5 \bar{x}^2 + S_6 \bar{x}^3 = 0$$

If we replace t_6 by a variable parameter t, and put σ's for the elementary symmetrical combinations of $t_1 \cdots t_5$, we have

$$S_1 = \sigma_1 + t, \qquad S_2 = \sigma_2 + t\sigma_1,$$
$$S_3 = \sigma_3 + t\sigma_2, \qquad S_4 = \sigma_4 + t\sigma_3,$$
$$S_5 = \sigma_5 + t\sigma_4, \qquad S_6 = t\sigma_5 .$$

If we make this substitution we get

$$x^3 - (\sigma_1 + t)x^2 + (\sigma_2 + t\sigma_1)x - (\sigma_3 + t\sigma_2) + (\sigma_4 + t\sigma_3)\bar{x} - (\sigma_5 + t\sigma_4)\bar{x}^2 + \sigma_5 t \bar{x}^3 = 0.$$

This is the equation of a pencil of orthic cubics through five points of a circle.

The centre of the curve through six points is $x = \frac{1}{3} S_1$. If the sixth point moves around the unit circle, this becomes

read

$$x = \frac{1}{3}(\sigma_1 + t)$$

read

This is the map equation of a circle . We have thus the

TP theorem: The locus of centres of the orthic cubics through five points of a circle is a circle. Its radius is one third that of the given circle, and its centre is the point $\frac{1}{3}\sigma_1$

M. Serret[1] gives an elegant synthetic proof of the theorem that the locus of centres of the curves of a pencil of equilaterals is a circle. I obtained the same result for orthic curves independently, and as the above analysis is so direct, it seems advisable to let it stand.

IV.

I shall now prove, for the pencil of orthic cubics through five points of a circle, a theorem, which M. Serret[1] states without proof. The theorem referred to, when stated for orthic cubics of the pencil under discussion, becomes:

<u>The curve enveloped by the asymptotes of all the orthic cubics through five points of a circle is an hypocycloid[2] of order six and class five.</u> It is circumscribed to the centre circle of the pencil and its cusps lie on a concentric circle five times as large.

We found that the equation of an asymptote, in terms of the six points where the curve cuts the unit circle, is

$$(x - \tfrac{1}{3}s_1) + \sqrt[3]{s_6}\left(\bar{x} - \tfrac{1}{3}\tfrac{s_5}{s_6}\right) = 0.$$

If we replace t_6 by the parameter t, this becomes

$$x - \tfrac{1}{3}(\sigma_1 + t) + \sqrt[3]{\sigma_5 t}\left(\bar{x} - \tfrac{1}{3}\left\{\tfrac{\sigma_4}{\sigma_5} + \tfrac{1}{t}\right\}\right) = 0.$$

We seek the curve enveloped by this line, as t runs around the unit circle.

For the sake of simplicity, let us refer this equation

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(1) Sur les faisceaux reguliers et les equilateres d'ordre n. Paul Serret, Comptes Rendus, 1895. t.121.. p., 373, 375.

(2) Figure Two

to a new system of coordinates, so chosen that the centre circle of the pencil becomes the new unit circle. The equation becomes

$$x - t + \sqrt[3]{\sigma_5 t}\left(\bar{x} - \frac{1}{t}\right) = 0$$

If now we take an axis of reals which makes $\sigma_5 = 1$ and and also put τ^3 for t, we have

$$x\tau^{-1} - \tau^2 + \bar{x} - \tau^{-3} = 0 .$$

The map equation of the curve enveloped by this line is obtained by equating to zero the result of differentiating with respect to τ. It is

$$x = 3\tau^{-2} - 2\tau^3 .$$

This is a curve of double circular motion. The curve is of order six, for it meets any line,

$$x = \frac{a}{1-\tau} ,$$

where

$$\frac{a}{1-\tau} = \frac{3}{\tau^2} - 2\tau^3$$

Or

$$2\tau^6 - 2\tau^5 - a\tau^2 + 3\tau - 3 = 0$$

This gives six τ's, and therefore the curve is of the sixth order. In order to determine the class of the curve, we must examine the equation of a tangent.

$$x\tau^{-1} - \tau^2 + \bar{x} - \tau^{-3} = 0$$

This is of the fifth degree in the parameter and there are, therefore, five tangents from any point x

The stationary points, or cusps, are the points where

the velocity of x is zero. For such a point, we must have $\mathfrak{D}_\tau x = 0$, and ~~is the point is~~ at the same time $|\tau| =$

Both these conditions are satisfied by

$$\tau = \sqrt[5]{-1}.$$

The curve has, therefore, five real cusps; one when τ is ~~one~~ each of the fifth roots of minus one.

If we put $\kappa^5 = -1$, we get a cusp,

$$x = 3\kappa^{-2} - 2\kappa^3$$

$$\kappa^2 x = 5$$

Since multiplication by κ^2 is equivalent to a rotation $\frac{4\pi}{5}$ we see that the locus of cusps is a circle, about the centre of the pencil, and five times as large as the centre circle. A rotation $\frac{2\pi}{5}$ sends each cusp into another and so the cusps are equally spaced along the cusp circle. The intersections of the hypocycloid with the centre circle,

$$x\bar{x} = 1,$$

are obtained by solving $x = \bar{x}^{-1}$ for τ.

We have

$$x = 3\tau^{-2} - 2\tau^3,$$

and

$$\bar{x} = 3\tau^2 - 2\tau^{-3}.$$

The parameters of the points sought are the roots

$$12\tau^5 - 6\tau^{10} - 6 = 0$$

or

$$(\tau^5 - 1)^2 = 0$$

There are five pairs of coincident intersections. But since x cannot be less than 1, it follows that the curve is tangent to the circle in five places.

We have obtained this hypocycloid as the locus of one asymptote. But all three asymptotes envelope the same curve, for, if we put ω for $\sqrt[3]{\sigma_5}$, we get

$$x = 3\omega\tau^{-2} - 2\tau^3$$

This has a cusp at $\kappa^4 x = 5$; it is, obviously, the same curve.

V.

The equation of a tangent to the curve is

$$x\tau^{-1} - \tau^2 + \bar{x} - \tau^{-3} = 0.$$

That of a perpendicular tangent.

$$-x\tau^{-1} - \tau^2 + \bar{x} + \tau^{-3} = 0.$$

These two lines meet at

$$x = \tau^{-2}.$$

In other words, Perpendicular tangents to the envelope of the asymptotes meet on the centre circle.

We have here a verification of the known property of the hypocycloid of this class that the tangents from a a point of the vertex circle are all real, and form two regular pencils.[1]

[1] F. Morley. On the Epicycloid. American Journal of Mathematics, Vol XIII, No 2.

VI .

Let us now consider the figure of nine orthocentric points, five of which are on a circle. The equation of the pencil of orthic cubics through five points of a circle is

$$x^3 - (\sigma_1 + t)\, x^2 + (\sigma_2 + t\,\sigma_1)\, x - (\sigma_3 + t\,\sigma_2)$$
$$+ (\sigma_4 + t\,\sigma_3)\,\bar{x} - (\sigma_5 + t\,\sigma_4)\,\bar{x}^2 + t\,\sigma_5\,\bar{x}^3 = 0.$$

We know five of the points of the orthocentric nine-point determined by this ~~circle~~ pencil, and we seek the remaining four. Rewrite the above equation as

$$x(x^2 - \sigma_1 x + \sigma_2) - t(x^2 - \sigma_1 x + \sigma_2)$$
$$-(\sigma_3 - \sigma_4 \bar{x} + \sigma_5 \bar{x}^2) + t\,\bar{x}(\sigma_3 - \sigma_4 \bar{x} + \sigma_5 \bar{x}^2) = 0.$$

or

$$(x - t)(x^2 - \sigma_1 x + \sigma_2) + (t\,\bar{x} - 1)(\sigma_3 - \sigma_4 \bar{x} + \sigma_5 \bar{x}^2) = 0$$

Now if both

$$x^2 - \sigma_1 x + \sigma_2,$$

and

$$\bar{x}^2 \sigma_5 - \sigma_4 \bar{x} + \sigma_3,$$

can become zero for conjugate values of x and $\bar{x}$, then ~~then~~ those values are the coordinates of a real point ~~of~~ which is on every curve of the pencil, and is one of the nine points. If we put $\sigma_5 = 1$, as we may, these two relations become

$$x^2 - \sigma_1 x + \sigma_2 = 0.$$

and

$$\bar{x}^2 - \bar{\sigma}_1 \bar{x} + \bar{\sigma}_2 = 0.$$

These are conjugate equations and so can be satisfied by the coordinates of real points. Solving them we get a pair of real points:

$$x_1 = \frac{\sigma_1 + \sqrt{\sigma_1^2 - 4\sigma_2}}{2}, \quad \bar{x}_1 = \frac{\bar{\sigma}_1 + \sqrt{\bar{\sigma}_1^2 - 4\bar{\sigma}_2}}{2};$$

and

$$x_2 = \frac{\sigma_1 - \sqrt{\sigma_1^2 - 4\sigma_2}}{2}, \quad \bar{x}_2 = \frac{\bar{\sigma}_1 - \sqrt{\bar{\sigma}_1^2 - 4\bar{\sigma}_2}}{2}$$

But further, we notice that the anti points, $x_1, \bar{x}_2$, and $x_2, \bar{x}_1$, of these make the equation of the pencil vanish for all values of the parameter. They are the remaining points of the orthocentric nine. This leads to the theorem that:

<u>If five points of an orthocentric nine-point are on a circle, of the remaining four points, two are real, two are imaginary; and these four form a central orthocentric four-point.</u>

The centroid of the nine points is

$$x = \frac{t_1 + t_2 + \cdots + t_5 + 2\sigma_1}{9} = \frac{1}{3}\sigma_1$$

This is the centre of the centre-circle of the pencil.

We can extend these results to the case of n^2 points, 2n-1 of which lie on a circle.

The pencil of orthic curves of order n which go through 2n-1 points of the unit circle is given by

$$x^n - (\sigma_1 + t)x^{n-1} + (\sigma_2 + t\sigma_1)x^{n-2} + - \cdots$$

$$+ (\sigma_{2n-2} + t\sigma_{2n-3})\bar{x}^{n-2} - (\sigma_{2n-1} + t\sigma_{2n-2})\bar{x}^{n-1} + \sigma_{2n-1} t\,\bar{x}^n = 0.$$

If we let $\sigma_{2n-1} = 1$, this becomes

$$(x-t)(x^{n-1} - \sigma_1 x^{n-2} + \sigma_2 x^{n-3} + - + \cdots \pm \sigma_{n-1})$$

$$+ (\bar{x}t - 1)(\bar{x}^{n-1} - \bar{\sigma}_1 \bar{x}^{n-2} + - + \cdots \pm \bar{\sigma}_{n-1}) = 0.$$

Now since the coefficients of $(x - t)$ and $(\bar{x}t - 1)$ are conjugate forms, there are n - 1 real points, in addition to the points $t_1, t_2 \cdots t_{2n-1}$, which are on all the curves of the pencil. Further, all the anti-points obtained by pairing these in all posible ways satisfy the equati n for all values of t. Now we know that the $(n-1)^2$ points thus found form an orthocentric set. We are now in a position to state the following general theorem.

<u>If 2n - 1 points of an orthocentric set of n^2 points lie on a circle, then the remaining $(n-1)^2$ points of the figure form a central orthocentric set of which</u> n - 1 <u>points are real.</u>

The x's of the n-1 real points are the roots of

$$x^{n-1} - \sigma_1 x^{n-2} + \sigma_2 x^{n-3} - + \cdots \sigma_{n-1} = 0.$$

VII .

We are now ready to consider the most general pencil of orthic curves. Form the equation

$$x^n - (a_1 + t a'_1)x^{n-1} + - \cdots$$
$$- (a_{2n-1} + t\, a'_{2n-1})\bar{x}^{n-1} + t\bar{x}^n = 0,$$

where t is a parameter which has the absolute value unity. Now for every value of t this represents a real orthic curve of the n^{th} order, provided

$$\overline{a_r + t\, a'_r} = a_{2n-r}t^{-1} + a'_{2n-r},$$

or

$$[\bar{a}_r - a'_{2n-r}] = [\bar{a}'_r - a_{2n-r}]t^{-1}$$

For if this holds, the equation can be put in the known form

$$(x - \alpha_1)(x - \alpha_2)\cdots = \tau_1(\bar{x} - \bar{\alpha}_1)(\bar{x} - \bar{\alpha}_2)\cdots$$

Now let

$$x^n - \alpha_1 x^{n-1} + \cdots \alpha_n \cdots - \alpha_{2n-1}\bar{x}^{n-1} + \alpha_{2n}\bar{x}^n = 0,$$

and

$$x^n - \alpha'_1 x^{n-1} + \cdots \alpha'_n \cdots - \alpha'_{2n-1}\bar{x}^{n-1} + \alpha'_{2n}\bar{x}^n = 0,$$

be the equations of any two real orthic curves. Then

$$\alpha_{2n} = t_1, \quad \alpha'_{2n} = t_2,$$

and

$$\bar{\alpha}_r = \alpha_{2n-r}\cdot t_1^{-1}, \quad \bar{\alpha}'_r = \alpha'_{2n-r}\cdot t_2^{-1}.$$

We can choose the a's in such a way that the pencil will include the given curves, (1) and (2), for the 4n-2 equations

$$\alpha_r = a_r + t_1 a'_r,$$
$$\alpha'_r = a_r + t_2 a'_r, \qquad r = 1 \cdots 2n-1$$

just suffice. We must show now that when the coefficients are determined as above, all the curves of the pencil are real.

Now we have

$$\alpha_r = a_r + t_1 a'_r,$$

and

$$\alpha_{2n-r} = a_{2n-r} + t_1 a'_{2n-r}.$$

From these, we get

$$\overline{a_r + t_1 a'_r} = \overline{\alpha}_r = \alpha_{2n-r} \cdot t_1^{-1} = a_{2n-r} t_1^{-1} + a'_{2n-r},$$

and therefore

$$\left[\overline{a}_r - a'_{2n-r}\right] = \left[\overline{a}'_r - a_{2n-r}\right] t_1^{-1}$$

But this is the condition that every curve of the pencil be real. It is clear that no curve not orthic can be included in the pencil. So we see that :

<u>Any two real orthic curves of order n determine a pencil of real curves of the same order, all of which are orthic.</u>

VIII.

M. Serret's theorem (Part Three, IV) on the locus of centres is easily verified. The centre of any curve of the pencil is

$$x = \frac{1}{n}\left(a_1 + t' a_1'\right).$$

Now if t is regarded as a parameter, this is the map equation of a circle with its centre at

$$x = \frac{1}{n} a_1$$

<u>The locus of centres of the most general pencil of orthic curves is a circle.</u>

In the special case where n of the intersections of the pencil are at infinity, the locus of centres degenerates into a right line. A pencil of this type may be written

$$x^n - \frac{a_1 - \lambda a_1'}{1 - \lambda} x^{n-1} + - \cdots \frac{a_{2n-1} - \lambda a'_{2n-1}}{1 - \lambda} \bar{x}^{n-1} + \bar{x}^n = 0,$$

where λ is a real parameter. The locus of centres is

$$x = \frac{1}{n} \cdot \frac{a_1 - \lambda a_1'}{1 - \lambda}$$

The elimination of λ from this and its conjugate gives

$$x\left(a_{2n-1} - a'_{2n-1}\right) - \bar{x}\left(a_1 - a_1'\right) + \frac{1}{n}\left(a_1 a'_{2n-1} - a_1' a_{2n-1}\right) = 0,$$

the equation of a right line.

IX.

Let us now seek the curve enveloped by the asymptotes of the curves of a general pencil. The equation of an asymptote of the curve given by t_1 is

$$x - \frac{1}{n}(a_1 + t_1 a_1') + \sqrt[n]{t_1}\left\{\bar{x} - \frac{1}{n}(a_{2n-1} t_1^{-1} + a'_{2n-1})\right\} = 0,$$

or

$$x - \frac{1}{n}(a_1 + t_1 a_1') + \sqrt[n]{t_1}\left\{\bar{x} - \frac{1}{n}(\bar{a}_1 - \bar{a}_1' t^{-1})\right\} = 0.$$

For convenience, transform to the centre of the pencil, $\frac{a_1}{n}$, as a new origin. The equation becomes

$$x - \frac{1}{n}a_1' t_1 + \sqrt[n]{t_1}\,(\bar{x} - \frac{1}{n}\bar{a}_1' t_1^{-1}) = 0$$

Putting $\tau^n = t$, we get

$$x - \frac{1}{n}a_1'\tau^n + \tau\bar{x} - \frac{1}{n}\bar{a}_1'\tau^{1-n} = 0,$$

and finally,

$$x\tau^{-1} - \frac{1}{n}a_1'\tau^{n-1} + \bar{x} - \frac{1}{n}\bar{a}_1'\tau^{-n} = 0.$$

Now the map equation of the curve enveloped by this line as τ varies is

$$-x\tau^{-2} - \frac{n-1}{n}a_1'\tau^{n-2} + \bar{a}_1'\tau^{-n-1} = 0,$$

or

$$nx = n\bar{a}_1'\tau^{1-n} + (1-n)a_1'\tau^n$$

Now this equation represents a curve of double circular motion. We know that

$$\bar{a}_1' = a_1' t_2^{-1},$$

and using it we get

$$nx = n a_1' t_2^{-1}\tau^{1-n} + (1-n)a_1'\tau^n.$$

Now if we make t_2 real, and then regard the centre circle as the unit circle, i.e., adopt $\left|\frac{a_1}{n}\right|$ as the unit length,

the equation takes the form

$$x = n\tau^{1-n} + (1-n)\tau^{n}.$$

This is the equation of an hypocycloid of the kind found as the locus of asymptotes of a special pencil of orthic cubics. Its vertex circle is the centre circle of the pencil. It has cusps when

$$D_\tau x = 0,$$

and

$$|\tau| = 1,$$

simultaneously, or when

$$\tau^{2n-1} + 1 = 0.$$

The parameters of the cusps are the $2n-1^{st}$ roots of -1.

If we let $\kappa^{2n-1} = -1$, a cusp is

$$x = n\kappa^{1-n} + (1-n)\kappa^{n}$$

or

$$x\kappa^{n-1} = n - (1-n)$$

The absolute value of a cusp is, therefore, $2n-1$.

Since the equation of a tangent,

$$x - \frac{1}{n}a_1'\tau^{n} + \tau\bar{x} - \frac{1}{n}\tau^{1-n}\bar{a}_1' = 0,$$

is of the $2n-1^{st}$ degree in the parameter τ, the hypocycloid is of class $2n-1$. It is of order $2n$, for if we eliminate x between the equation of the curve and the equation of any line.

$$\lambda = \frac{a}{1-\tau}$$

we get an equation of the $2n^{th}$ degree to determine the parameters of the points of intersection. The curve meets any line in 2n points, and is therefore of order 2n. We have now established analytically the theorem stated by M. Serret, as far as orthic curves are concerned. It is:

The curve enveloped by the asymptotes of a pencil of orthic curves of order n is an hypocycloid of order 2n, and of class 2n-1. Its vertex circle is the centre circle of the pencil, and its ~~cusps~~ cusp circle is concentric with that circle, and 2n-1 times as large.

If we bear in mind that any difference between an orthic curve and any equilateral does not affect the the terms of the n^{th} and $n-1^{st}$ degrees of the equation, we see that the method of proof used above is applicable to equilaterals in general.

X .

It is a well known proposition that the centres of the equilateral hyperbolas circumscribed to a triangle lie on the circle through the mid-points of the sides of the triangle. This circle is usually called the Feuerbach, or nine-point, circle of the triangle. Now we have seen

that an orthic curve of order n may be made to satisfy $2n$ linear conditions; it follows that any odd number, $2n-1$, of points determines a pencil of orthic curves of the n^{th} order. Connected with this pencil is the center-circle, or, as I propose to call it, the Serret circle, which is in a sense, the generalized nine-point circle.

Every figure of an odd number of points has connected with it a unique circle; The Serret circle, which in the case of three points, is identical with the nine-point circle of Feuerbach.

Further, every odd number of points, $2n-1$, determine the pencil of orthic curves through them, and therefore the remaining $(n-1)^2$ points of the orthocentric n^2-point. In the case of three given points, this set of $(n-1)^2$ points is a single point, the orthocentre of the given points. So we are lead to the theorem:

To every figure of $2n-1$ points belongs a figure of $(n-1)^2$ points.

In one sense, the Serret circle belongs to n2 points, but of these, only $2n-1$ may be taken at random.

XI.

Now consider an even number, $2n$, of points which do

not belong to an orthocentric n^2-point. There is a pencil of orthic curves through every 2n-1 points which can be selected from them, or 2n pencils in all. Now these pencils give rise to 2n- Serret circles, but there is one orthic curve through all 2n points and its centre is on each of the circles. We have, therefore, the result:

The 2n Serret circles, given by all the sets of 2n-1 among 2n points, meet in a point.

XII.

In section VIII., we obtained the pencil of orthic curves determined by the two given curves,

(1). $$x^n - x_1 x^{n-1} + x_2 x^{n-2} + \cdots x_{2n-1}\bar{x}^{n-1} + x_{2n}\bar{x}^n = 0,$$

and

(2). $$x^n - x_1' x^{n-1} + x_2' x^{n-2} + \cdots x_{2n-1}'\bar{x}^{n-1} + x_{2n}'\bar{x}^n = 0.$$

We now wish to show that the centroid of the orthocentric n^2-point in which these two curves intersect is the centre of the centre circle of the pencil. If we rewrite (1) and (2) in terms of $\bar{x}$ we get

(1). $$(\bar{x} - \bar{x}_1)(\bar{x} - \bar{x}_2)\cdots(\bar{x} - \bar{x}_n) = 0,$$

and

(2) $$(\bar{x} - \bar{x}_1')(\bar{x} - \bar{x}_2')\cdots(\bar{x} - \bar{x}_n') = 0.$$

If the S's refer to the symmetrical functions of the roots. We have

$$S_1 = \alpha_{2n-1},\quad S_1' = \alpha_{2n-1}',\ \cdots\ S_{n-1} = \alpha_{n+1},$$
$$S_n = -(x^n - \alpha_1 x^{n-1} + \alpha_2 x^{n-2} \cdots \alpha_n)\alpha_{2n}^{-1}$$
$$S_n' = -(x^n - \alpha_1' x^{n-1} + \alpha_2' x^{n-2} \cdots \alpha_n')\alpha_{2n}'^{-1} \qquad (1) \text{ and } (2),$$

Now the eliminant of x between these two equations is

$$(\bar{x}_1 - \bar{x}_1')(\bar{x}_1 - \bar{x}_2') \cdots (\bar{x}_1 - \bar{x}_n')\cdot$$
$$(\bar{x}_2 - \bar{x}_1')(\bar{x}_2 - \bar{x}_2') \cdots (\bar{x}_2 - \bar{x}_n')\cdot$$
$$\cdots\cdots\cdots\cdots$$
$$(\bar{x}_n - \bar{x}_1')(\bar{x}_n - \bar{x}_2') \cdots (\bar{x}_n - \bar{x}_n') = 0.$$

This is a function of degree n^2 in x, and as x occurs in S_n and S_n' alone, we need consider only those terms in which the products S_n and S_n' appear. These are:

$$S_n^n - nS_n^{n-1}S_n' + \frac{n \cdot n-1}{2} S_n^{n-2} S_n'^2 \cdots \pm S_n'^n = 0,$$

or

$$(S_n - S_n')^n;$$

or, in terms of x,

$$\left\{\left(\frac{\alpha_{2n} - \alpha_{2n}'}{\alpha_{2n}'\alpha_{2n}}\right)x^n - \left(\frac{\alpha_{2n}\alpha_1' - \alpha_{2n}'\alpha_1}{\alpha_{2n}'\alpha_{2n}}\right)x^{n-1} + \cdots \left(\frac{\alpha_{2n}\alpha_n' - \alpha_{2n}'\alpha_n}{\alpha_{2n}'\alpha_{2n}}\right)\right\}^n.$$

When this is expanded and arranged in powers of x, the first and second terms are

$$x^{n^2}\left(\frac{\alpha_{2n} - \alpha_{2n}'}{\alpha_{2n}'\alpha_{2n}}\right)^n - n\left(\frac{\alpha_{2n} - \alpha_{2n}'}{\alpha_{2n}'\alpha_{2n}}\right)^{n-1}\left(\frac{\alpha_{2n}\alpha_1' - \alpha_{2n}'\alpha_1}{\alpha_{2n}'\alpha_{2n}}\right)x^{n^2-1}.$$

Now the sum of the roots is

$$\sigma_1 = n\,\frac{\alpha_{2n}\alpha_1' - \alpha_{2n}'\alpha_1}{\alpha_{2n} - \alpha_{2n}'},$$

and their centroid is

$$\frac{1}{n^2}\sigma_1 = \frac{1}{n}\cdot\frac{\alpha_{2n}\alpha_1' - \alpha_{2n}'\alpha_1}{\alpha_{2n} - \alpha_{2n}'} = x'$$

Now $\alpha_{2n} = t_1$, and $\alpha'_{2n} = t_2$, and also,

We have the relations

$$\alpha_r = a_r + t_1 a'_r,$$

and

$$\alpha'_r = a_r + t_2 a'_r,$$

from which we obtain

$$x' = \frac{1}{n}\,\frac{t_1 a_1 + t_1 t_2 a'_1 - t_2 a_1 - t_2 t_1 a'_1}{t_1 - t_2}$$

$$= \frac{1}{n} a_1 .$$

But this is precisely the centre of the centre-circle

$$x = \frac{1}{n}(a_1 + t a'_1).$$

We are thus enabled to conclude with the general theorem:

<u>The centroid of an orthocentric set of points is the centre of the centre-circle of the pencil of orthic curves through those points.</u>

Biographical Note.

I, Charles Edward Brooks, was born in Baltimore, August twenty-sixth, 1879. I received my preparation for college at the University School for Boys, in Baltimore. I matriculated in the Johns Hopkins University in October, 1897. I followed the Mathematical-Physical group of studies, and proceeded to the degree of Bachelor of Arts in June, 1900. Since that time, I have been a graduate student of Mathematics, Philosophy, and Physics in this University.